UNIQUENESS THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS WITH HÖLDER CONTINUITY

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ABSTRACT. We establish some uniqueness results near 0 for ordinary differential equations of the type $u^{(n)}(t) = f(u(t))$ which satisfies $u(0) = u'(0) = \cdots = u^{(m-1)}(0) = 0$ and $u^{(m)}(0) \neq 0$ with $m \geq n$. We also show that although f is not Lipschitz near 0, it is always Hölder continuous.

1. INTRODUCTION

A standard theorem in ODE theory states that in the equation $u'(t) = F(t, x(t))$, if the function $F(t, x)$ is Lipschitz continuous with respect to variable x in an interval containing the initial point, then the initial value problem has a unique solution locally. Generally speaking, with equations of arbitrary order, to obtain the uniqueness of solution we need to assume a Lipschitz condition on F. In this paper, we study autonomous ODEs of the type $u^{(n)}(t) = f(u(t))$ where $u \in C^\infty([0,1])$ satisfies the initial conditions $u(0) = u'(0) = \cdots = u^{(m-1)}(0) = 0$ and $u^{(m)}(0) = a \neq 0$, where $m \geq n$. That is, the order of the lowest non-vanishing derivative of u at 0 is no less than the order of the equation. For this type of equations, it is not difficult to see that f is differentiable away from 0, however, it is not differentiable at 0. In fact, f is not Lipschitz near 0. For example,

Example 1. *Let $u(t) = t^3$, then $u(0) = u'(0) = u''(0) = 0$, $u^{(3)}(0) = 6$, and u satisfies*

$$u''(t) = 6u^{\frac{1}{3}}.$$

The function $f(u) = 6u^{\frac{1}{3}}$ is not Lipschitz near 0.

Interestingly, although f is not Lipschitz near 0, we can prove that it is always Hölder continuous, as summarized in Theorem 1.5 below.

Due to the lack of Lipschitz continuity, the standard uniqueness theory no longer applies to this type of equations. We will show that if u has vanishing derivatives at 0 up to certain order, then such solutions are unique near 0. Specifically, we have the following result.

Theorem 1.1. *Let $u(t) \in C^\infty([0,1))$ be a solution of the differential equation*

$$u^{(n)}(t) = f(u(t)), \tag{1}$$

where $n \geq 1$ and f is a function. Assume that u satisfies

$$u(0) = u'(0) = \cdots = u^{(m-1)}(0) = 0 \text{ and } u^{(m)}(0) = a \neq 0 \tag{2}$$

with $m \geq n$. Then such a solution $u(t)$ is unique for t near 0.

Typically, for an n-th order ODE we need only n initial conditions. This theorem shows that in some sense, the lack of Lipschitz continuity can be compensated by assuming additional derivative information at the initial point.

On the other hand, if u vanishes to infinite order at 0, the result will not hold either.

Example 2. *The function*

$$u(t)=\begin{cases} e^{-\frac{1}{t}} & 0<t\leq 1\\ 0 & t=0\end{cases}$$

is in $C^{\infty}([0,1])$ *and*

$$u^{(k)}(0)=0 \qquad \textit{for all} \quad k\in\mathbf{N}. \tag{3}$$

Let

$$f(s)=\begin{cases} s(\ln s)^2 & s>0\\ 0 & s=0\end{cases}$$

Then $u(t)$ *satisfies the equation*

$$u'=f(u).$$

However, this equation has another solution $u\equiv 0$*, which also satisfies (3).*

Furthermore, Theorem 1.1 can be strengthened to the following:

Theorem 1.2. *Suppose* u_1 *and* u_2 *are two solutions of Equation (1) and they satisfy*

$$u_1(0) = u_1'(0) = \cdots = u_1^{(m-1)}(0) = 0, \quad u_1^{(m)}(0)=a\neq 0 \tag{4}$$

and

$$u_2(0) = u_2'(0) = \cdots = u_2^{(l-1)}(0) = 0, \quad u_2^{(l)}(0)=b\neq 0 \tag{5}$$

where $m,l\geq n$*. Then* $m=l$*,* $a=b$ *and* $u_1\equiv u_2$ *for small* t*.*

The proof of Theorem 1.1 is carried out in two steps. First, we show the following result concerning the derivatives of u at 0.

Lemma 1.3. *Let* $u(t)$ *be a solution that satisfies Equations (1) and (2). The derivative of* u *at 0 of any order equal to or higher than m, that is,* $u^{(k)}(0)$ *for any* $k\geq m$*, depends only on* m,n*, and the behavior of the function* f *near 0.*

In the second step, we make use of a unique continuation theorem in [2].

Theorem 1.4. *(*[2]*) Let* $g(x)\in C^{\infty}([a,b])$*,* $0\in[a,b]$*, and*

$$|g^{(n)}(x)|\leq C\sum_{k=1}^{n-1}\frac{\left|g^{(k)}(x)\right|}{|x|^{n-k}}, \qquad x\in[a,b] \tag{6}$$

for some constant C *and some* $n\geq 1$*. Then*

$$g^{(k)}(0)=0 \quad \forall k\geq 0$$

implies

$$g\equiv 0 \ \textit{ on } [a,b]$$

The proof of Lemma 1.3 will be given in Section 2, and the proof of Theorems 1.1 and 1.2 will be given in Section 3.

If a function $u(t)$ satisfies Condition (2), then as shown in Section 2, locally t can be expressed as a function of u, therefore we can express $u^{(n)}(t)$ locally as a function f of u, so $u^{(n)}(t) = f(u)$. The next theorem shows that the function f is Hölder continuous in an interval $[0, \delta]$ for small $\delta > 0$.

Theorem 1.5. *Suppose a function $u(t)$ satisfies Condition (2), then Equation (1) holds for some function f where $m \geq n$, and there is a constant $\delta > 0$ such that f is uniformly Hölder continuous in the interval* $[0, \delta]$.

This theorem is proved after Theorem 1.2 in Section 3.

A summary of Theorems 1.1 and 1.5 is that any smooth function of finite order vanishing at 0 is a unique local solution of a differential equation in the form of (1), where f is differentiable in the interior and uniformly Hölder continuous up to the boundary.

We note that in [1] Li and Nirenberg studied a similar second order PDE: $\Delta u = f(u)$, where $u = u(t, x) \in C^\infty(\mathbf{R}^{k+1})$ has a non-vanishing partial derivative at 0 which can be expressed in the form $u(t, x) = at^m + O(t^{m+1})$, $a \neq 0$, $t \in \mathbf{R}$ and $x \in \mathbf{R}^k$. They showed that if two solutions u and v satisfies $u \geq v$, then $u \equiv v$. Theorem 1.1 can be viewed as an improvement of their result in the one-dimensional case to arbitrary order and without the comparison condition $u \geq v$.

2. The Proof of Lemma 1.3

Without loss of generality, we can assume that $a > 0$.

- First, we show that a, the m-th derivative of u at 0, only depends on m, n, and the function f.

Define

$$\tilde{x} = \left(\frac{u}{a}\right)^{\frac{1}{m}}. \tag{7}$$

Then

$$\tilde{x} \;=\; \left(t^m + O\left(t^{m+1}\right)\right)^{\frac{1}{m}} \;=\; t\left(1 + O(t)\right).$$

This implies that

$$\frac{\tilde{x}}{t} \to 1 \quad \text{as} \quad t \to 0. \tag{8}$$

We can also write $u = a\tilde{x}^m$. Taking the derivative with respect to t, we get

$$\begin{aligned}
\frac{du}{dt} &= am\tilde{x}^{m-1}\frac{d\tilde{x}}{dt} \\
amt^{m-1} + O(t^m) &= am\tilde{x}^{m-1}\frac{d\tilde{x}}{dt} \\
\frac{t^{m-1}}{\tilde{x}^{m-1}} + O\left(\frac{t^m}{\tilde{x}^{m-1}}\right) &= \frac{d\tilde{x}}{dt}
\end{aligned}$$

In the second equation above and in the analysis that follows, we formally differentiate the Taylor expansion with the big-O notation. A detailed discussion of this differentiation is provided in the Appendix.

In light of (8), it follows that

$$\frac{d\tilde{x}}{dt}|_{t=0} = 1. \tag{9}$$

By the Inverse Function Theorem, t can be expressed as a function of $\tilde{x}$: $t = \tilde{x} + O\left(\tilde{x}^2\right)$. Then

$$t^{m-n} = \left(\tilde{x} + O\left(\tilde{x}^2\right)\right)^{m-n} = \tilde{x}^{m-n}\left(1 + O\left(\tilde{x}\right)\right).$$

Similarly, $t^{m-n+1} = \tilde{x}^{m-n+1}\left(1 + O\left(\tilde{x}\right)\right)$. Thus

$$\begin{aligned}
f(u) &= u^{(n)} \\
&= am(m-1)\cdots(m-n+1)t^{m-n} + O\left(t^{m-n+1}\right) \\
&= am(m-1)\cdots(m-n+1)\tilde{x}^{m-n} + O\left(\tilde{x}^{m-n+1}\right) \\
&= am(m-1)\cdots(m-n+1)\left(\frac{u}{a}\right)^{\frac{m-n}{m}} + O\left(u^{\frac{m-n+1}{m}}\right) \qquad \text{by Equation (7)} \\
&= a^{\frac{n}{m}}m(m-1)\cdots(m-n+1)u^{\frac{m-n}{m}} + O\left(u^{\frac{m-n+1}{m}}\right) \qquad (10)
\end{aligned}$$

Therefore,

$$a^{\frac{n}{m}} = \lim_{u\to 0}\frac{f(u)}{m(m-1)\cdots(m-n+1)u^{\frac{m-n}{m}}}. \tag{11}$$

This shows that a is completely determined by m, n, and the behavior of the function f near 0.

- Next, we show that the $(m+1)$-th derivative of u at 0 also only depends on m, n, and f.

Write u as

$$u(t) = at^m + a_{m+1}t^{m+1} + O(t^{m+2}). \tag{12}$$

We will show that a_{m+1} only depends on m, n, and the behavior of f at 0.
Express t as

$$t = \tilde{x} + b_2\tilde{x}^2 + O\left(\tilde{x}^3\right). \tag{13}$$

We would like to obtain an expression for b_2 in terms of the derivatives of u at 0. To do this, we take the derivative with respect to t on both sides of $\frac{d\tilde{x}}{dt} \cdot \frac{dt}{d\tilde{x}} = 1$. By the Product Rule and Chain Rule, we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{d\tilde{x}}{dt}\right) \cdot \frac{dt}{d\tilde{x}} + \frac{d\tilde{x}}{dt} \cdot \frac{d}{dt}\left(\frac{dt}{d\tilde{x}}\right) &= 0 \\ \frac{d^2\tilde{x}}{dt^2} \cdot \frac{dt}{d\tilde{x}} + \frac{d\tilde{x}}{dt} \cdot \left(\frac{d^2t}{d\tilde{x}^2} \cdot \frac{d\tilde{x}}{dt}\right) &= 0 \\ \frac{d^2\tilde{x}}{dt^2} \cdot \frac{dt}{d\tilde{x}} + \left(\frac{d\tilde{x}}{dt}\right)^2 \cdot \frac{d^2t}{d\tilde{x}^2} &= 0 \end{aligned} \tag{14}$$

We would like to evaluate Equation (14) at $t = 0$.
From

$$\begin{aligned} \tilde{x} &= \left(\frac{u}{a}\right)^{\frac{1}{m}} \\ &= \left(t^m + \frac{a_{m+1}}{a}t^{m+1} + O\left(t^{m+2}\right)\right)^{\frac{1}{m}} \\ &= t\left[1 + \frac{a_{m+1}}{a}t + O(t^2)\right]^{\frac{1}{m}} \\ &= t\left[1 + \frac{1}{m}\left(\frac{a_{m+1}}{a}t + O(t^2)\right) + \frac{1}{2} \cdot \frac{1}{m}\left(\frac{1}{m} - 1\right)\left(\frac{a_{m+1}}{a}t + O(t^2)\right)^2 + O(t^3)\right] \\ &= t + \frac{a_{m+1}}{ma}t^2 + O(t^3), \end{aligned}$$

we know that

$$\frac{d^2\tilde{x}}{dt^2}|_{t=0} = \frac{2a_{m+1}}{ma}. \tag{15}$$

From (13) we know that

$$\frac{dt}{d\tilde{x}}|_{t=0} = 1 \qquad \text{and} \qquad \frac{d^2t}{d\tilde{x}^2}|_{t=0} = 2b_2.$$

Thus if we evaluate Equation (14) at $t = 0$, we get

$$\frac{2a_{m+1}}{ma} \cdot 1 + 1 \cdot 2b_2 = 0,$$

therefore

$$b_2 = -\frac{a_{m+1}}{ma}. \tag{16}$$

Now, from Equation (13) we have

$$\begin{aligned}
t^{m-n} &= \left(\tilde{x}+b_2\tilde{x}^2+O\left(\tilde{x}^3\right)\right)^{m-n} \\
&= \tilde{x}^{m-n}\left[1+b_2\tilde{x}+O\left(\tilde{x}^2\right)\right]^{m-n} \\
&= \tilde{x}^{m-n}\left[1+(m-n)\left(b_2\tilde{x}+O\left(\tilde{x}^2\right)\right)+O\left(\tilde{x}^2\right)\right] \\
&= \tilde{x}^{m-n}+(m-n)b_2\tilde{x}^{m-n+1}+O\left(\tilde{x}^{m-n+2}\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
t^{m-n+1} &= \tilde{x}^{m-n+1}+(m-n+1)b_2\tilde{x}^{m-n+2}+O\left(\tilde{x}^{m-n+3}\right). \\
t^{m-n+2} &= O\left(\tilde{x}^{m-n+2}\right).
\end{aligned}$$

Then from Equation (12) and the above expressions for the powers of t, we have

$$\begin{aligned}
u^{(n)} &= am(m-1)\cdots(m-n+1)t^{m-n} \\
&\quad +a_{m+1}(m+1)m\cdots(m-n+2)t^{m-n+1}+O\left(t^{m-n+2}\right) \\
&= am(m-1)\cdots(m-n+1)\left[\tilde{x}^{m-n}+(m-n)b_2\tilde{x}^{m-n+1}+O\left(\tilde{x}^{m-n+2}\right)\right] \\
&\quad +a_{m+1}(m+1)m\cdots(m-n+2)\left[\tilde{x}^{m-n+1}+(m-n+1)b_2\tilde{x}^{m-n+2}+O\left(\tilde{x}^{m-n+3}\right)\right] \\
&\quad +O\left(\tilde{x}^{m-n+2}\right) \\
&= am(m-1)\cdots(m-n+1)\tilde{x}^{m-n} \\
&\quad +\left[am(m-1)\cdots(m-n)b_2+a_{m+1}(m+1)m\cdots(m-n+2)\right]\tilde{x}^{m-n+1}+O\left(\tilde{x}^{m-n+2}\right)
\end{aligned}$$

Thus by $f(u)=u^{(n)}$ and (7), we have

$$\begin{aligned}
f(u) &= am(m-1)\cdots(m-n+1)\tilde{x}^{m-n} \\
&\quad +\Big[am(m-1)\cdots(m-n)b_2+a_{m+1}(m+1)m\cdots(m-n+2)\Big]\tilde{x}^{m-n+1}+O\left(\tilde{x}^{m-n+2}\right) \\
&= am(m-1)\cdots(m-n+1)\left(\frac{u}{a}\right)^{\frac{m-n}{m}} \\
&\quad +\Big[am(m-1)\cdots(m-n)b_2+a_{m+1}(m+1)m\cdots(m-n+2)\Big]\left(\frac{u}{a}\right)^{\frac{m-n+1}{m}} \\
&\quad +O\left(\left(\frac{u}{a}\right)^{\frac{m-n+2}{m}}\right) \\
&= a^{\frac{n}{m}}m(m-1)\cdots(m-n+1)u^{\frac{m-n}{m}} \\
&\quad +\Big[a^{\frac{n-1}{m}}m(m-1)\cdots(m-n)b_2+a^{\frac{n-m-1}{m}}a_{m+1}(m+1)m\cdots(m-n+2)\Big]u^{\frac{m-n+1}{m}} \\
&\quad +O\left(u^{\frac{m-n+2}{m}}\right)
\end{aligned}$$

This means that

$$\begin{aligned}
& a^{\frac{n-1}{m}}m(m-1)\cdots(m-n)b_2+a^{\frac{n-m-1}{m}}a_{m+1}(m+1)m\cdots(m-n+2) \\
=\ & \lim_{u\to 0}\frac{f(u)-a^{\frac{n}{m}}m(m-1)\cdots(m-n+1)u^{\frac{m-n}{m}}}{u^{\frac{m-n+1}{m}}}.
\end{aligned}$$

By (16), this can be written as

$$\begin{aligned} & a^{\frac{n-1}{m}} m(m-1)\cdots(m-n)\left(-\frac{a_{m+1}}{ma}\right) + a^{\frac{n-m-1}{m}} a_{m+1}(m+1)m\cdots(m-n+2) \\ = \ & \lim_{u\to 0} \frac{f(u) - a^{\frac{n}{m}} m(m-1)\cdots(m-n+1)u^{\frac{m-n}{m}}}{u^{\frac{m-n+1}{m}}}. \end{aligned}$$

After collecting similar terms, we get

$$\begin{aligned} & a_{m+1} a^{\frac{n-m-1}{m}} \left[(m+1)m\cdots(m-n+2) - (m-1)(m-2)\cdots(m-n)\right] \\ = \ & \lim_{u\to 0} \frac{f(u) - a^{\frac{n}{m}} m(m-1)\cdots(m-n+1)u^{\frac{m-n}{m}}}{u^{\frac{m-n+1}{m}}}. \end{aligned}$$

Consequently,

$$a_{m+1} = \frac{a^{\frac{m-n+1}{m}} \cdot \left(\lim\limits_{u\to 0} \dfrac{f(u) - a^{\frac{n}{m}} m(m-1)\cdots(m-n+1)u^{\frac{m-n}{m}}}{u^{\frac{m-n+1}{m}}}\right)}{(m+1)m\cdots(m-n+2) - (m-1)(m-2)\cdots(m-n)}. \tag{17}$$

Since we have proved that a only depends on m, n, and f, Equation (17) shows that a_{m+1} is also completed determined by m, n, and the behavior of f near 0. By (16), this also shows that b_2 depends only on m, n, and f.

- Then, we will use mathematical induction to show that all the derivatives of u at 0 of order higher than m are completely determined by m, n, and f.

Express u and t as

$$u(t) = at^m + a_{m+1}t^{m+1} + \cdots + a_{m+k}t^{m+k} + a_{m+k+1}t^{m+k+1} + O\left(t^{m+k+2}\right), \tag{18}$$

$$t = \tilde{x} + b_2\tilde{x}^2 + \cdots + b_{k+1}\tilde{x}^{k+1} + b_{k+2}\tilde{x}^{k+2} + O\left(\tilde{x}^{k+3}\right) \tag{19}$$

Suppose that for $k \geq 1$, a, a_{m+1}, ..., a_{m+k}, b_2,..., b_{k+1} are all determined only by by m, n, and f, we will show that a_{m+k+1} and b_{k+2} also are determined only by by m, n, and f.

We start by obtaining an expression for b_{k+2} in terms of a_{m+1}, ..., a_{m+k} and a_{m+k+1}.

Taking the derivative with respect to t on both sides of (14), we obtain

$$\begin{aligned} 0 &= \frac{d^3\tilde{x}}{dt^3}\cdot\frac{dt}{d\tilde{x}} + \frac{d^2\tilde{x}}{dt^2}\cdot\left(\frac{d^2t}{d\tilde{x}^2}\cdot\frac{d\tilde{x}}{dt}\right) + 2\cdot\frac{d\tilde{x}}{dt}\cdot\frac{d^2\tilde{x}}{dt^2}\cdot\frac{d^2t}{d\tilde{x}^2} + \left(\frac{d\tilde{x}}{dt}\right)^2\cdot\left(\frac{d^3t}{d\tilde{x}^3}\cdot\frac{d\tilde{x}}{dt}\right) \\ 0 &= \frac{d^3\tilde{x}}{dt^3}\cdot\frac{dt}{d\tilde{x}} + 3\cdot\frac{d\tilde{x}}{dt}\cdot\frac{d^2\tilde{x}}{dt^2}\cdot\frac{d^2t}{d\tilde{x}^2} + \left(\frac{d\tilde{x}}{dt}\right)^3\cdot\frac{d^3t}{d\tilde{x}^3} \qquad (20) \end{aligned}$$

Taking the derivative of both sides of (20), we get

$$
\begin{aligned}
0 \;=\;& \left(\frac{d^4\tilde{x}}{dt^4}\cdot\frac{dt}{d\tilde{x}}+\frac{d^3\tilde{x}}{dt^3}\cdot\frac{d^2t}{d\tilde{x}^2}\cdot\frac{d\tilde{x}}{dt}\right)+3\left(\frac{d^2\tilde{x}}{dt^2}\cdot\frac{d^2\tilde{x}}{dt^2}\cdot\frac{d^2t}{d\tilde{x}^2}+\frac{d\tilde{x}}{dt}\cdot\frac{d^3\tilde{x}}{dt^3}\cdot\frac{d^2t}{d\tilde{x}^2}\right.\\
&\left.+\frac{d\tilde{x}}{dt}\cdot\frac{d^2\tilde{x}}{dt^2}\cdot\frac{d^3t}{d\tilde{x}^3}\cdot\frac{d\tilde{x}}{dt}\right)+\left(3\left(\frac{d\tilde{x}}{dt}\right)^2\cdot\frac{d^2\tilde{x}}{dt^2}\cdot\frac{d^3t}{d\tilde{x}^3}+\left(\frac{d\tilde{x}}{dt}\right)^3\cdot\frac{d^4t}{d\tilde{x}^4}\cdot\frac{d\tilde{x}}{dt}\right)\\
0 \;=\;& \frac{d^4\tilde{x}}{dt^4}\cdot\frac{dt}{d\tilde{x}}+4\cdot\frac{d^3\tilde{x}}{dt^3}\cdot\frac{d\tilde{x}}{dt}\cdot\frac{d^2t}{d\tilde{x}^2}+3\cdot\frac{d^2\tilde{x}}{dt^2}\cdot\frac{d^2\tilde{x}}{dt^2}\cdot\frac{d^2t}{d\tilde{x}^2}\\
&+6\left(\frac{d\tilde{x}}{dt}\right)^2\cdot\frac{d^2\tilde{x}}{dt^2}\cdot\frac{d^3t}{d\tilde{x}^3}+\left(\frac{d\tilde{x}}{dt}\right)^4\cdot\frac{d^4t}{d\tilde{x}^4} \qquad (21)
\end{aligned}
$$

If we keep taking the derivative with respect to t for k times and collect the similar terms after each differentiation as shown above, eventually we will arrive at an expression of the form

$$
\begin{aligned}
0 \;=\;& \frac{d^{k+2}\tilde{x}}{dt^{k+2}}\cdot\frac{dt}{d\tilde{x}}+\left(\text{terms involving }\frac{d^{k+1}\tilde{x}}{dt^{k+1}},\frac{d^k\tilde{x}}{dt^k},\ldots,\frac{d\tilde{x}}{dt},\frac{dt}{d\tilde{x}},\frac{d^2t}{d\tilde{x}^2},\ldots,\frac{d^{k+1}t}{d\tilde{x}^{k+1}}\right)\\
&+\left(\frac{d\tilde{x}}{dt}\right)^{k+2}\cdot\frac{d^{k+2}t}{d\tilde{x}^{k+2}} \qquad (22)
\end{aligned}
$$

From (19) we know that

$$
\begin{aligned}
\frac{dt}{d\tilde{x}}|_{t=0} &= 1\\
\frac{d^2t}{d\tilde{x}^2}|_{t=0} &= 2b_2\\
\vdots &= \vdots \qquad (23)\\
\frac{d^{k+1}t}{d\tilde{x}^{k+1}}|_{t=0} &= (k+1)!b_{k+1}\\
\frac{d^{k+2}t}{d\tilde{x}^{k+2}}|_{t=0} &= (k+2)!b_{k+2}
\end{aligned}
$$

Then we look at $\frac{d\tilde{x}}{dt}|_{t=0}$, ... , $\frac{d^{k+1}\tilde{x}}{dt^{k+1}}|_{t=0}$, and $\frac{d^{k+2}\tilde{x}}{dt^{k+2}}|_{t=0}$.

By definition (7) and Equation (18),

$$
\begin{aligned}
\tilde{x} &= \left(\frac{at^m + a_{m+1}t^{m+1} + a_{m+2}t^{m+2} + \cdots + a_{m+k}t^{m+k} + a_{m+k+1}t^{m+k+1} + O\left(t^{m+k+2}\right)}{a}\right)^{\frac{1}{m}} \\
&= t\left(1 + \frac{a_{m+1}}{a}t + \frac{a_{m+2}}{a}t^2 + \cdots + \frac{a_{m+k}}{a}t^k + \frac{a_{m+k+1}}{a}t^{k+1} + O\left(t^{k+2}\right)\right)^{\frac{1}{m}} \\
&= t\Bigg\{1 + \frac{1}{m}\left[\frac{a_{m+1}}{a}t + \frac{a_{m+2}}{a}t^2 + \cdots + \frac{a_{m+k}}{a}t^k + \frac{a_{m+k+1}}{a}t^{k+1} + O\left(t^{k+2}\right)\right] \\
&\quad + \frac{1}{2}\cdot\frac{1}{m}\left(\frac{1}{m}-1\right)\left[\frac{a_{m+1}}{a}t + \frac{a_{m+2}}{a}t^2 + \cdots + \frac{a_{m+k}}{a}t^k + \frac{a_{m+k+1}}{a}t^{k+1} + O\left(t^{k+2}\right)\right]^2 \\
&\quad + \cdots \\
&\quad + \frac{1}{(k+1)!}\cdot\frac{1}{m}\cdot\left(\frac{1}{m}-1\right)\cdots\left(\frac{1}{m}-k\right)\left[\frac{a_{m+1}}{a}t + \frac{a_{m+2}}{a}t^2 + \cdots + \frac{a_{m+k}}{a}t^k\right. \\
&\quad \left. + \frac{a_{m+k+1}}{a}t^{k+1} + O\left(t^{k+2}\right)\right]^{k+1} + O\left(t^{k+2}\right)\Bigg\}
\end{aligned}
$$

After collecting similar terms we can write

$$\tilde{x} = t + \lambda_2 t^2 + \lambda_3 t^3 + \cdots + \lambda_{k+1}t^{k+1} + \left(\frac{a_{m+k+1}}{ma} + \lambda_{k+2}\right)t^{k+2} + O\left(t^{k+3}\right), \tag{24}$$

where

- λ_2 is a constant involving m, a, and a_{m+1}
- λ_3 is a constant involving m, a, a_{m+1} and a_{m+2}

$\vdots$

- λ_{k+1} is a constant involving $m, a, a_{m+1}, ..., a_{m+k-1}, a_{m+k}$
- λ_{k+2} is a constant involving $m, a, a_{m+1}, ..., a_{m+k-1}, a_{m+k}$

By the inductive hypothesis, $\lambda_2, \lambda_3, ...\lambda_{k+1}, \lambda_{k+2}$ are all constants that only depend on m, n, and the function f.

From Equation (24) we obtain

$$
\begin{aligned}
\frac{d\tilde{x}}{dt}|_{t=0} &= 1 \\
\frac{d^2\tilde{x}}{dt^2}|_{t=0} &= 2\lambda_2 \\
\vdots &= \vdots \\
\frac{d^{k+1}\tilde{x}}{dt^{k+1}}|_{t=0} &= (k+1)!\lambda_{k+1} \\
\frac{d^{k+2}\tilde{x}}{dt^{k+2}}|_{t=0} &= (k+2)!\left(\frac{a_{m+k+1}}{ma} + \lambda_{k+2}\right)
\end{aligned} \tag{25}
$$

Now we evaluate (22) at $t = 0$ and make use of (23) and (25):

$$0=(k+2)!\left(\frac{a_{m+k+1}}{ma}+\lambda_{k+2}\right)\cdot 1+\left(\text{terms involving } b_2,\dots,b_{k+1},\lambda_2,\dots,\lambda_{k+1}\right)+1\cdot(k+2)!b_{k+2}.$$

Thus we obtain

$$b_{k+2}=-\frac{a_{m+k+1}}{ma}+Q, \tag{26}$$

where Q is a constant depending on $b_2,\dots,b_{k+1},\lambda_2,\dots,\lambda_{k+1},\lambda_{k+2}$, hence Q is completely determined by m,n, and f.

Next we will analyze a_{m+k+1}. From (19) we have

$$\begin{aligned} t^{m-n} &= \tilde{x}^{m-n}\left(1+b_2\tilde{x}+\cdots+b_{k+1}\tilde{x}^k+b_{k+2}\tilde{x}^{k+1}+O\left(\tilde{x}^{k+2}\right)\right]^{m-n} \\ &= \tilde{x}^{m-n}\Big\{1+(m-n)\left[b_2\tilde{x}+\cdots+b_{k+1}\tilde{x}^k+b_{k+2}\tilde{x}^{k+1}+O\left(\tilde{x}^{k+2}\right)\right] \\ &\quad+\frac{(m-n)(m-n-1)}{2}\left[b_2\tilde{x}+\cdots+b_{k+1}\tilde{x}^k+b_{k+2}\tilde{x}^{k+1}+O\left(\tilde{x}^{k+2}\right)\right]^2+\cdots \\ &\quad+\frac{(m-n)(m-n-1)\cdots(m-n-k)}{(k+1)!}\left[b_2\tilde{x}+\cdots+b_{k+1}\tilde{x}^k+b_{k+2}\tilde{x}^{k+1}+O\left(\tilde{x}^{k+2}\right)\right]^{k+1} \\ &\quad+O\left(\tilde{x}^{k+2}\right)\Big\} \end{aligned}$$

After collecting similar terms we can express t^{m-n} as

$$\begin{aligned} t^{m-n} &= \tilde{x}^{m-n}\Big\{1+c_{1,m-n}\tilde{x}+c_{2,m-n}\tilde{x}^2+\cdots+c_{k,m-n}\tilde{x}^k \\ &\quad+\left[(m-n)b_{k+2}+c_{k+1,m-n}\right]\tilde{x}^{k+1}+O\left(\tilde{x}^{k+2}\right)\Big\}, \end{aligned}$$

where $c_{1,m-n}$ is a constant depending on m and b_2; $c_{2,m-n}$ is a constant depending on m,b_2 and b_3; ... ; $c_{k,m-n}$ is a constant depending on $m,b_2,\dots,b_{k+1}$; $c_{k+1,m-n}$ is a constant depending on $m,b_2,\dots,b_{k+1}$.

By the inductive hypothesis, $c_{1,m-n},c_{2,m-n},\dots,c_{k,m-n}$ and $c_{k+1,m-n}$ are all determined only by m,n, and f. Thus we have

$$\begin{aligned} t^{m-n} &= \tilde{x}^{m-n}+c_{1,m-n}\tilde{x}^{m-n+1}+c_{2,m-n}\tilde{x}^{m-n+2}+\cdots+c_{k,m-n}\tilde{x}^{m-n+k} \\ &\quad+\left[(m-n)b_{k+2}+c_{k+1,m-n}\right]\tilde{x}^{m-n+k+1}+O\left(\tilde{x}^{m-n+k+2}\right), \end{aligned} \tag{27}$$

where $c_{1,m-n},c_{2,m-n},\dots,c_{k,m-n}$ and $c_{k+1,m-n}$ are constants depending on m,n, and f.

By the same type of analysis we obtain similar expressions for the other powers of t:

$$\begin{aligned} t^{m-n+1} &= \tilde{x}^{m-n+1}+c_{1,m-n+1}\tilde{x}^{m-n+2}+c_{2,m-n+1}\tilde{x}^{m-n+3}+\cdots+c_{k,m-n+1}\tilde{x}^{m-n+k+1} \\ &\quad+\left[(m-n+1)b_{k+2}+c_{k+1,m-n+1}\right]\tilde{x}^{m-n+k+2}+O\left(\tilde{x}^{m-n+k+3}\right), \end{aligned} \tag{28}$$

where $c_{1,m-n+1}, c_{2,m-n+1}, \ldots, c_{k,m-n+1}$ and $c_{k+1,m-n+1}$ are constants depending on m, n, and f.

$$\begin{aligned} t^{m-n+2} &= \tilde{x}^{m-n+2} + c_{1,m-n+2}\tilde{x}^{m-n+3} + c_{2,m-n+2}\tilde{x}^{m-n+4} + \cdots + c_{k,m-n+2}\tilde{x}^{m-n+k+2} \\ &\quad + \left[(m-n+2)b_{k+2} + c_{k+1,m-n+2}\right]\tilde{x}^{m-n+k+3} + O\left(\tilde{x}^{m-n+k+4}\right), \end{aligned} \tag{29}$$

where $c_{1,m-n+2}, c_{2,m-n+2}, \ldots, c_{k,m-n+2}$ and $c_{k+1,m-n+2}$ are constants depending on m, n, and f.

$\vdots$

$$\begin{aligned} t^{m-n+k} &= \tilde{x}^{m-n+k} + c_{1,m-n+k}\tilde{x}^{m-n+k+1} + c_{2,m-n+k}\tilde{x}^{m-n+k+2} + \cdots + c_{k,m-n+k}\tilde{x}^{m-n+2k} \\ &\quad + \left[(m-n+k)b_{k+2} + c_{k+1,m-n+k}\right]\tilde{x}^{m-n+2k+1} + O\left(\tilde{x}^{m-n+2k+2}\right), \end{aligned} \tag{30}$$

where $c_{1,m-n+k}, c_{2,m-n+k}, \ldots, c_{k,m-n+k}$ and $c_{k+1,m-n+k}$ are constants depending on m, n, and f.

$$\begin{aligned} t^{m-n+k+1} &= \tilde{x}^{m-n+k+1} + c_{1,m-n+k+1}\tilde{x}^{m-n+k+2} \\ &\quad + c_{2,m-n+k+1}\tilde{x}^{m-n+k+3} + \cdots + c_{k,m-n+k+1}\tilde{x}^{m-n+2k+1} \\ &\quad + \left[(m-n+k+1)b_{k+2} + c_{k+1,m-n+k+1}\right]\tilde{x}^{m-n+2k+2} + O\left(\tilde{x}^{m-n+2k+3}\right), \end{aligned} \tag{31}$$

where $c_{1,m-n+k+1}, c_{2,m-n+k+1}, \ldots, c_{k,m-n+k+1}$ and $c_{k+1,m-n+k+1}$ are constants depending on m, n, and f.

$$t^{m-n+k+2} = O\left(\tilde{x}^{m-n+k+2}\right) \tag{32}$$

From (18) we obtain

$$\begin{aligned} u^{(n)} &= am(m-1)\cdots(m-n+1)t^{m-n} + a_{m+1}(m+1)m\cdots(m-n+2)t^{m-n+1} \\ &\quad + \cdots + a_{m+k}(m+k)(m+k-1)\cdots(m-n+k+1)t^{m-n+k} \\ &\quad + a_{m+k+1}(m+k+1)(m+k)\cdots(m-n+k+2)t^{m-n+k+1} + O\left(t^{m-n+k+2}\right) \end{aligned}$$

Then by (27) to (32) we can write

$$\begin{aligned}
u^{(n)} &= am(m-1)\cdots(m-n+1)\Big\{\tilde{x}^{m-n} + c_{1,m-n}\tilde{x}^{m-n+1} + c_{2,m-n}\tilde{x}^{m-n+2} + \cdots \\
&\quad + c_{k,m-n}\tilde{x}^{m-n+k} + \left[(m-n)b_{k+2} + c_{k+1,m-n}\right]\tilde{x}^{m-n+k+1} + O\left(\tilde{x}^{m-n+k+2}\right)\Big\} \\
&\quad + a_{m+1}(m+1)m\cdots(m-n+2)\Big\{\tilde{x}^{m-n+1} + c_{1,m-n+1}\tilde{x}^{m-n+2} + c_{2,m-n+1}\tilde{x}^{m-n+3} \\
&\quad + \cdots + c_{k,m-n+1}\tilde{x}^{m-n+k+1} + \left[(m-n+1)b_{k+2} + c_{k+1,m-n+1}\right]\tilde{x}^{m-n+k+2} \\
&\quad + O\left(\tilde{x}^{m-n+k+3}\right)\Big\} + \cdots \\
&\quad + a_{m+k}(m+k)(m+k-1)\cdots(m+k-n+1)\Big\{\tilde{x}^{m-n+k} + c_{1,m-n+k}\tilde{x}^{m-n+k+1} \\
&\quad + c_{2,m-n+k}\tilde{x}^{m-n+k+2} + \cdots + c_{k,m-n+k}\tilde{x}^{m-n+2k} + \\
&\quad \left[(m-n+k)b_{k+2} + c_{k+1,m-n+k}\right]\tilde{x}^{m-n+2k+1} + O\left(\tilde{x}^{m-n+2k+2}\right)\Big\} \\
&\quad + a_{m+k+1}(m+k+1)(m+k)\cdots(m+k-n+2)\Big\{\tilde{x}^{m-n+k+1} + \\
&\quad c_{1,m-n+k+1}\tilde{x}^{m-n+k+2} + c_{2,m-n+k+1}\tilde{x}^{m-n+k+3} + \cdots + c_{k,m-n+k+1}\tilde{x}^{m-n+2k+1} \\
&\quad + \left[(m-n+k+1)b_{k+2} + c_{k+1,m-n+k+1}\right]\tilde{x}^{m-n+2k+2} + O\left(\tilde{x}^{m-n+2k+3}\right)\Big\} \\
&\quad + O\left(\tilde{x}^{m-n+k+2}\right) \\
&= am(m-1)\cdots(m-n+1)\tilde{x}^{m-n} + C\left(m, a, a_{m+1}, c_{1,m-n}\right)\tilde{x}^{m-n+1} \\
&\quad + C\left(m, a, a_{m+1}, a_{m+2}, c_{1,m-n+1}, c_{2,m-n}\right)\tilde{x}^{m-n+2} + \cdots \\
&\quad + C\left(m, a, a_{m+1}, ..., a_{m+k}, c_{k,m-n}, c_{k-1,m-n+1}, ..., c_{1,m-n+k-1}\right)\tilde{x}^{m-n+k} \\
&\quad + \Big[am(m-1)\cdots(m-n)b_{k+2} + a_{m+k+1}(m+k+1)(m+k)\cdots(m+k-n+2) \\
&\quad + C\left(m, a, a_{m+1}, ..., a_{m+k}, c_{k+1,m-n}, c_{k,m-n+1}, ..., c_{1,m-n+k}\right)\Big]\tilde{x}^{m-n+k+1} \\
&\quad + O\left(\tilde{x}^{m-n+k+2}\right).
\end{aligned}$$

Here $C\left(m, a, a_{m+1}, c_{1,m-n}\right)$ is a constant depending on m, a, a_{m+1} and $c_{1,m-n}$; we denote it as p_{m-n+1} to simplify notations. Since a, a_{m+1} and $c_{1,m-n}$ only depend on m, n, and f, we know that p_{m-n+1} only depends on m, n, and f.

Similarly, the other constants $C\left(m, a, a_{m+1}, a_{m+2}, c_{1,m-n+1}, c_{2,m-n}\right)$, ..., and $C\left(m, a, a_{m+1}, ..., a_{m+k}, c_{k+1,m-n}, c_{k,m-n+1}, ..., c_{1,m-n+k}\right)$ all depend only on m, n, and f, and can be denoted simply as p_{m-n+2},..., p_{m-n+k}, and $p_{m-n+k+1}$. Thus we can rewrite the above equation as

$$\begin{aligned}
u^{(n)} &= am(m-1)\cdots(m-n+1)\tilde{x}^{m-n} + p_{m-n+1}\tilde{x}^{m-n+1} + \\
&\quad p_{m-n+2}\tilde{x}^{m-n+2} + \cdots + p_{m-n+k}\tilde{x}^{m-n+k} + \\
&\quad \Big[am(m-1)\cdots(m-n)b_{k+2} + a_{m+k+1}(m+k+1)(m+k)\cdots(m+k-n+2) \\
&\quad + p_{m-n+k+1}\Big]\tilde{x}^{m-n+k+1} + O\left(\tilde{x}^{m-n+k+2}\right) \qquad (33)
\end{aligned}$$

Now because of $u^{(n)} = f(u)$ and definition (7) , we have

$$\begin{aligned} f(u) \;=\;& am(m-1)\cdots(m-n+1)\left(\frac{u}{a}\right)^{\frac{m-n}{m}} + p_{m-n+1}\left(\frac{u}{a}\right)^{\frac{m-n+1}{m}} + \\ & p_{m-n+2}\left(\frac{u}{a}\right)^{\frac{m-n+2}{m}} + \cdots + p_{m-n+k}\left(\frac{u}{a}\right)^{\frac{m-n+k}{m}} + \\ & \Big[am(m-1)\cdots(m-n)b_{k+2} + a_{m+k+1}(m+k+1)(m+k)\cdots(m+k-n+2) \\ & + p_{m-n+k+1}\Big]\left(\frac{u}{a}\right)^{\frac{m-n+k+1}{m}} + O\left(u^{\frac{m-n+k+2}{m}}\right) \end{aligned}$$

Due to (26), we can rewrite the above equation as

$$\begin{aligned} f(u) \;=\;& am(m-1)\cdots(m-n+1)\left(\frac{u}{a}\right)^{\frac{m-n}{m}} + p_{m-n+1}\left(\frac{u}{a}\right)^{\frac{m-n+1}{m}} + \\ & p_{m-n+2}\left(\frac{u}{a}\right)^{\frac{m-n+2}{m}} + \cdots + p_{m-n+k}\left(\frac{u}{a}\right)^{\frac{m-n+k}{m}} + \\ & + \Big\{ \Big[(m+k+1)(m+k)\cdots(m+k-n+2) - (m-1)(m-2)\cdots(m-n)\Big] a_{m+k+1} \\ & + am(m-1)\cdots(m-n)Q + p_{m-n+k+1} \Big\}\left(\frac{u}{a}\right)^{\frac{m-n+k+1}{m}} + O\left(u^{\frac{m-n+k+2}{m}}\right) \end{aligned} \tag{34}$$

From (34) we get

$$\begin{aligned} & \Big[(m+k+1)(m+k)\cdots(m+k-n+2) - (m-1)(m-2)\cdots(m-n)\Big] a_{m+k+1} \\ & + am(m-1)\cdots(m-n)Q + p_{m-n+k+1} \\ = \;& \lim_{u\to 0} \frac{f(u) - am\cdots(m-n+1)\left(\frac{u}{a}\right)^{\frac{m-n}{m}} - p_{m-n+1}\left(\frac{u}{a}\right)^{\frac{m-n+1}{m}} - \cdots - p_{m-n+k}\left(\frac{u}{a}\right)^{\frac{m-n+k}{m}}}{\left(\frac{u}{a}\right)^{\frac{m-n+k+1}{m}}} \end{aligned}$$

Note that $(m+k+1)(m+k)\cdots(m+k-n+2) - (m-1)(m-2)\cdots(m-n) \neq 0$, then since the constants $Q, a, p_{m-n+1}, \ldots, p_{m-n+k+1}$ all depend only on m, n, and f, we know that a_{m+k+1} only depends on m, n, and f. Consequently, b_{k+2} also only depends on m, n, and f because of (26).

Therefore, by mathematical induction, all derivatives of f at 0 are determined completely by m, n, and f. This completes the proof of Lemma 1.3.

3. The Proofs of Theorems 1.1, 1.2 and 1.5

Theorem 1.1

Proof: Suppose there are two solutions $u(t)$ and $v(t)$, both satisfy Equations (1) and (2). By Lemma 1.3, at $t = 0$, u and v have the same derivative of any order. Let $w = u - v$, then

$$w^{(k)}(0) = 0 \quad \text{for any integer } \; k \geq 0.$$

In order to apply Theorem 1.4 we need to show that w satisfies Condition (6).

$$w^{(n)}(t) \;=\; u^{(n)}(t) - v^{(n)}(t) \;=\; f(u(t)) - f(v(t))$$

Without loss of generality we assume $a > 0$. By Equation (10), we can write

$$f(u) = \left[a^{\frac{n}{m}} m(m-1)\cdots(m-n+1)u^{\frac{m-n}{m}} + \alpha(u)\right] \tag{35}$$

and

$$f(v) = \left[a^{\frac{n}{m}} m(m-1)\cdots(m-n+1)v^{\frac{m-n}{m}} + \alpha(v)\right],$$

where α is a function with the order

$$\alpha(s) = O\left(s^{\frac{m-n+1}{m}}\right).$$

So we can write

$$w^{(n)}(t) = a^{\frac{n}{m}} m(m-1)\cdots(m-n+1)\left(u^{\frac{m-n}{m}} - v^{\frac{m-n}{m}}\right) + (\alpha(u) - \alpha(v)) \tag{36}$$

If $m = n$, then

$$a^{\frac{n}{m}} m(m-1)\cdots(m-n+1)\left(u^{\frac{m-n}{m}} - v^{\frac{m-n}{m}}\right) = 0$$

If $m > n$, by the Mean Value Theorem,

$$\left|u^{\frac{m-n}{m}} - v^{\frac{m-n}{m}}\right| \le \frac{m-n}{m}\zeta^{-\frac{n}{m}}|u - v| \tag{37}$$

where $\zeta(t)$ is between $u(t)$ and $v(t)$. Since $u(t) = at^m + O\left(t^{m+1}\right)$ and $v(t) = at^m + O\left(t^{m+1}\right)$, we know that $\zeta(t) = at^m + O\left(t^{m+1}\right)$, which implies

$$\zeta^{\frac{n}{m}} \;=\; a^{\frac{n}{m}} t^n\left(1 + O(t)\right) \;\ge\; Ct^n$$

for some constant $C > 0$ when t is sufficiently small. Thus

$$\zeta^{-\frac{n}{m}}|u - v| \le C^{-1}\frac{|u - v|}{t^n}$$

and by equation (37) we know that

$$a^{\frac{n}{m}} m(m-1)\cdots(m-n+1)\left|u^{\frac{m-n}{m}} - v^{\frac{m-n}{m}}\right| \le C\frac{|u - v|}{t^n} \tag{38}$$

for another constant $C > 0$.

Next, we estimate $|\alpha(u) - \alpha(v)|$.

From Equation (1), we know that f is differentiable with respect to t, since $u^{(n)}(t)$ is differentiable with respect to t. Condition (2) shows that $\frac{du}{dt}(t) \neq 0$ when $t > 0$ is sufficiently small. Then by the Inverse Function Theorem, t is differentiable with respect to u. Thus, when u is small and positive, f is differentiable with respect to u and

$$\frac{df}{du} = \frac{df}{dt}\cdot\frac{dt}{du}.$$

Then by Equation (35), since f is differentiable on a small interval $(0, \delta)$, α is also differentiable on a small interval $(0, \delta)$. By the Mean Value Theorem

$$\alpha(u) - \alpha(v) = \alpha'(\eta)(u - v)$$

where $\eta(t)$ is between $u(t)$ and $v(t)$. Because $u(t) = at^m + O\left(t^{m+1}\right)$ and $v(t) = at^m + O\left(t^{m+1}\right)$, we know that $\eta(t) = at^m + O\left(t^{m+1}\right) \geq Ct^m$ when t is small, thus

$$\eta^{-\frac{n-1}{m}} = O\left(t^{-(n-1)}\right).$$

From $\alpha(s) = O\left(s^{\frac{m-n+1}{m}}\right)$ we get $\alpha'(s) = O\left(s^{-\frac{n-1}{m}}\right)$. Therefore

$$\alpha'(\eta) \;=\; O\left(\eta^{-\frac{n-1}{m}}\right) \;=\; O\left(t^{-(n-1)}\right).$$

Thus for some $C > 0$

$$\begin{aligned} |\alpha(u) - \alpha(v)| &\leq Ct^{-(n-1)}|u - v| \\ &\leq Ct^{-n}|u - v| \quad \text{since } 0 < t < 1 \end{aligned} \tag{39}$$

Combining Equations (36), (38), and (39), we conclude

$$|w^{(n)}(t)| \;\leq\; C\frac{|u(t) - v(t)|}{t^n} \;=\; C\frac{|w(t)|}{t^n}.$$

Finally, extend the domain of $w(t)$ to $[-1, 1]$ by defining $w(t) = w(-t)$ when $-1 \leq t < 0$. Then $w \in C^\infty([-1, 1])$ and it satisfies Condition (6). By Theorem 1.4, $w \equiv 0$, which means $u \equiv v$.
This completes the proof of Theorem 1.1.

□

Theorem 1.2

Proof: Without loss of generality we assume $a > 0$. We apply the same analysis as in the proof of Lemma 1.3 to u_1 and u_2, respectively. Similar to (11) we have

$$a^{\frac{n}{m}} \;=\; \lim_{u_1 \to 0} \frac{f(u_1)}{m(m-1)\cdots(m-n+1)u_1^{\frac{m-n}{m}}} \;=\; \lim_{s \to 0} \frac{f(s)}{m(m-1)\cdots(m-n+1)s^{\frac{m-n}{m}}}$$

and

$$b^{\frac{n}{l}} \;=\; \lim_{u_2 \to 0} \frac{f(u_2)}{l(l-1)\cdots(l-n+1)u_2^{\frac{l-n}{l}}} \;=\; \lim_{s \to 0} \frac{f(s)}{l(l-1)\cdots(l-n+1)s^{\frac{l-n}{l}}}.$$

Suppose $m \neq l$, without loss of generality we assume $m < l$. Dividing the two equations, we get

$$\begin{aligned} a^{\frac{n}{m}} b^{-\frac{n}{l}} &= \frac{l(l-1)\cdots(l-n+1)}{m(m-1)\cdots(m-n+1)} \lim_{s \to 0} s^{\frac{l-n}{l} - \frac{m-n}{m}} \\ &= \frac{l(l-1)\cdots(l-n+1)}{m(m-1)\cdots(m-n+1)} \lim_{s \to 0} s^{\frac{n}{m} - \frac{n}{l}}. \end{aligned}$$

Since $m < l$, $\lim\limits_{s \to 0} s^{\frac{n}{m} - \frac{n}{l}} = 0$. However, $a^{\frac{n}{m}} b^{-\frac{n}{l}} \neq 0$. This is a contradiction.

Therefore $m = l$, consequently $a = b$. Then by Theorem 1.1 we know that $u_1 \equiv u_2$ for small t.

□

Theorem 1.5:

Proof: : The proof of Lemma 1.3 shows that near 0, t is a function of u, therefore $u^{(n)}(t)$ can be expressed as a function f of u. Thus Equation (1) holds when $t > 0$ is small. From Condition (2) we define $f(0) = 0$.

By the first two equations in (10) and the discussions in Appendix A about differentiating a Taylor expansion, we know that there is a function $h \in C^1([0, \epsilon])$ for some $\epsilon > 0$, such that

$$f(u) = am(m-1)\cdots(m-n+1)\tilde{x}^{m-n} + h(\tilde{x})\,\tilde{x}^{m-n}.$$

By definition (7) we have

$$f(u) = am(m-1)\cdots(m-n+1)\left(\frac{u}{a}\right)^{\frac{m-n}{m}} + h\left(\left(\frac{u}{a}\right)^{\frac{1}{m}}\right)\left(\frac{u}{a}\right)^{\frac{m-n}{m}} \tag{40}$$

Since $0 \le \frac{m-n}{m} < 1$ and $0 < \frac{1}{m} < 1$, it is well known that $u^{\frac{m-n}{m}}$ and $u^{\frac{1}{m}}$ are Hölder continuous on the closed interval $[0, 1]$ with Hölder coefficients $\frac{m-n}{m}$ and $\frac{1}{m}$, respectively. This implies that the first term in (40) is Hölder continuous on $[0, 1]$.

Since h is C^1 on $[0, \epsilon]$, it is also Hölder continuous on $[0, \epsilon]$. Then since the composition of two Hölder continuous functions is Hölder continuous, we know that $h\left(\left(\frac{u}{a}\right)^{\frac{1}{m}}\right)$ is Hölder continuous with respect to u on a closed interval $[0, \delta]$ with $\delta > 0$. Next, because the product of two Hölder continuous functions is also Hölder continuous, we know that $h\left(\left(\frac{u}{a}\right)^{\frac{1}{m}}\right) \cdot \left(\frac{u}{a}\right)^{\frac{m-n}{m}}$ is Hölder continuous. Thus the second term in (40) is Hölder continuous on $[0, \delta]$.

Therefore, f is Hölder continuous on $[0, \delta]$ and the theorem is proved.

□

Appendix A. Differentiation of the Taylor Expansion

We will discuss in detail the differentiation of the Taylor expansion of a function that is frequently used in the proof of Lemma 1.3.

In general, suppose $g(x) \in C^{k+1}([a, b])$, by the Taylor Theorem we can write

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots + \frac{g^{(k)}(a)}{k!}(x-a)^k + h(x)(x-a)^k \tag{41}$$

where $\lim_{x\to a} h(x) = 0$. An explicit expression for $h(x)$ is

$$h(x) = \frac{g^{(k+1)}(\xi)}{(k+1)!}(x-a) \tag{42}$$

where $a < \xi < x$. Usually we denote (41) as

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots + \frac{g^{(k)}(a)}{k!}(x-a)^k + O\left((x-a)^{k+1}\right). \tag{43}$$

Taking the derivative on both sides of Equation (41), we get

$$g'(x) = g'(a) + g''(a)(x-a) + \cdots + \frac{g^{(k)}(a)}{(k-1)!}(x-a)^{k-1} + h'(x)(x-a)^k + kh(x)(x-a)^{k-1}. \tag{44}$$

From (41) we know that $h(x)$ is C^1 on $(a, b]$. Next we show that it is actually C^1 up to the boundary, on $[a, b]$.

Define $h(a) = 0$, so h is continuous on $[a, b]$. Denote

$$P(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots + \frac{g^{(k)}(a)}{k!}(x-a)^k,$$

then

$$h(x) = \frac{g(x) - P(x)}{(x-a)^k}.$$

By the definition of limits,

$$\begin{aligned} h'(a) &= \lim_{x \to a} \frac{h(x) - h(a)}{x - a} \\ &= \lim_{x \to a} \frac{g(x) - P(x)}{(x-a)^{k+1}} \\ &\vdots \\ &= \frac{g^{(k+1)}(a) - P^{(k+1)}(a)}{(k+1)!} \qquad \text{by applying the L'Hospital Rule } k+1 \text{ times} \\ &= \frac{g^{(k+1)}(a)}{(k+1)!}, \end{aligned} \tag{45}$$

where we have used the fact that $P^{(k+1)}(a) = 0$.
When $x > a$,

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(\frac{g(x) - P(x)}{(x-a)^k}\right) \\ &= \frac{\left(g'(x) - P'(x)\right)(x-a)^k - (g(x) - P(x))k(x-a)^{k-1}}{(x-a)^{2k}} \\ &= \frac{g'(x) - P'(x)}{(x-a)^k} - \frac{k\left(g(x) - P(x)\right)}{(x-a)^{k+1}} \end{aligned}$$

By repeatedly applying the L'Hospital Rule, we know that

$$\lim_{x \to a} \frac{g'(x) - P'(x)}{(x-a)^k} = \frac{g^{(k+1)}(a) - P^{(k+1)}(a)}{k!} = \frac{g^{(k+1)}(a)}{k!}$$

and

$$\lim_{x \to a} \frac{k\left(g(x) - P(x)\right)}{(x-a)^{k+1}} = \frac{k\left(g^{(k+1)}(a) - P^{(k+1)}(a)\right)}{(k+1)!} = \frac{kg^{(k+1)}(a)}{(k+1)!}.$$

Therefore

$$\begin{aligned} \lim_{x \to a} h'(x) &= \frac{g^{(k+1)}(a)}{k!} - \frac{kg^{(k+1)}(a)}{(k+1)!} \\ &= \frac{g^{(k+1)}(a)}{(k+1)!} \end{aligned} \tag{46}$$

Equations (45) and (46) show that $h'(x)$ is C^1 on the closed interval $[a,b]$.

Furthermore, we know that for any $x \in [a,b]$, $|h'(x)| \leq C$ for some constant C_1, thus

$$\left|h'(x)(x-a)^k\right| \leq C_1 \left|x-a\right|^k .$$

Since $g(x) \in C^{k+1}([a,b])$, from (42) we know that $|h(x)| \leq C_2 \left|x-a\right|$ for some constant C_2, thus

$$\left|kh(x)(x-a)^{k-1}\right| \leq kC_2 \left|x-a\right|^k .$$

Therefore, (44) can be denoted as

$$g'(x) = g'(a) + g''(a)(x-a) + \cdots + \frac{g^{(k)}(a)}{(k-1)!}(x-a)^{k-1} + O\left(x-a\right)^k \tag{47}$$

Since the first, second, ... , and $(k-1)$-th derivatives of $g'(x)$ at a are $g''(a), g^{(3)}(a)$, ... , and $g^{(k)}(a)$, Equation (47) is the Taylor expansion of $g'(x)$ at a to order $k-1$. This shows that we can formally differentiate (43) to get (47).

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